Natural Deduction Systems for Intuitionistic Substructural Logics and their Strong Normalization(Theory of Rewriting Systems and Its Applications)

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数理解析研究所講究録
918 巻 1995 年 196-205

Natural Deduction Systems for Intuitionistic Substructural Logics and their Strong Normalization

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Abstract

An intuitionistic substructural logic is a formal system obtained from Gentzen's sequent calculus LJ for the intuitionistic logic by removing some or all of structural rules, i.e. *exchange*, *weakening* and *contraction*.

Some natural deduction systems for them are known, but they all are what we call 'sequent style' natural deduction. here we introduce 'pure' natural deduction systems, and consider their strong nomaliztion.

1 Introduction

An intuitionistic substructural logic is obtained from LJ by removing some or all of structural rules.

One of important properties in substructural logics is that the notion of *and* will split into two. The first one is called additive conjunction, and the other one is called multiplicative conjunction. Every natural deduction systems for substructural logics in literatures are restricted mostly to their multiplicative fragments, since adding additive connectives will cause much complications.

we will introduce natural deduction systems for four intuitionistic substructural logics with exchange rule containing both additive and multiplicative conjunctions, and prove their strong normalization theorem. We will show that when an intuitionistic substructural logic doesn't have contraction rule, an upper bound of the number of normalizing steps for a given proof Π can be easily calculated by Π. On the other hand, complicated arguments as in the strong normalization of the intuitionistic logic seem to be necessary for the intuitionistic substructural logic with contraction.

2 System ILL$*$

Here, we define four intuitionistic substractural logics defined by sequent calculus systems.

Definition 2.1 (formula) *Assume that there are finite or infinite propositional symbols. Then we define formulae as follows.*
All propositional symbols are formulae.
If A and B are formulae, then $(A \supset B)$, $(A \wedge B)$ and $(A * B)$ are formulae.

Definition 2.2 (System ILL) *System ILL is a sequent calculus system with cut and exchange rule and implication, multiplicative conjunction and additive conjuction fragments.*

Definition 2.3 (System ILL$-$W) *System ILL$-$W is a system obtained by adding a stractural rule* Weakening *to System ILL.*

Definition 2.4 (System ILL−C) *System ILL−C is a system obtained by adding a stractural rule* Contraction *to the System ILL.*

Definition 2.5 (System ILL−CW) *System ILL−CW is a formal system obtained by adding the stractural rule Weakening and Contraction to the system ILL.*

$$A \vdash A$$

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ Cut$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ Exchange$$

$$\frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, \Delta, A \supset B \vdash C} \supset L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \supset R$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L_l \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge L_r$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge R$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A * B \vdash C} *L \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A * B} *R$$

$$\frac{\Gamma \vdash B}{A, \Gamma \vdash B}\ Weakening$$

$$\frac{A, A, \Gamma \vdash B}{A, \Gamma \vdash B}\ Contraction$$

Rules for System ILL∗

3 System NILL∗

we consider natural deduction systems which are equivalent to System ILL∗.

Definition 3.1 (assumption) *If A is a formula, and if n is a natural number, then A^n is an assumption.*

Definition 3.2 (System NILL) *Let A, B and C be meta-variables for formulae, and let Γ and Δ be meta-variables for finite sets of assumptions. Then we define System NILL as follows:*

(assumption) *If A^n is an assumption, then A^n is a deduction proving $\{A^n\} \vdash A$.*

($\supset$ I) *If $\mathcal{D}$ is a deduction proving $\Gamma \vdash B$, and If (W−condition):$A^n \in \Gamma$ holds for, then*
$$\frac{\mathcal{D}}{A \supset B} (\supset I)^n$$
is a deduction proving $\Gamma - \{A^n\} \vdash A \supset B$.

($\supset$ E) *If $\mathcal{D}_1$ is a deduction proving $\Gamma \vdash A \supset B$, and $\mathcal{D}_2$ is a deduction proving $\Delta \vdash A$, and if (C−condition):$\Gamma \cap \Delta = \emptyset$ holds for, then*
$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{B} (\supset E)$$
is a deduction proving $\Gamma \cup \Delta \vdash B$.

($\wedge$ I) *If $\mathcal{D}_1$ is a deduction proving $\Gamma \vdash A$, and $\mathcal{D}_2$ is a deduction proving $\Delta \vdash B$, and if (W−condition):$\Gamma = \Delta$ holds for, then*
$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{A \wedge B} (\wedge I)$$
is a deduction proving $\Gamma \cup \Delta \vdash A \wedge B$.

($\wedge$ E$_l$) *If $\mathcal{D}_1$ is a deduction proving $\Gamma \vdash A \supset C$, and $\mathcal{D}_2$ is a deduction proving $\Delta \vdash A \wedge B$, and if (C−condition):$\Gamma \cap \Delta = \emptyset$ holds for, then*

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{C}\ (\wedge E_l)$$

is a deduction proving $\Gamma \cup \Delta \vdash C$.

($\wedge$ E$_r$) *If $\mathcal{D}_1$ is a deduction proving $\Gamma \vdash B \supset C$, and $\mathcal{D}_2$ is a deduction proving $\Delta \vdash A \wedge B$, and if (C−condition):$\Gamma \cap \Delta = \emptyset$ holds for, then*

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{C}\ (\wedge E_r)$$

is a deduction proving $\Gamma \cup \Delta \vdash C$.

($*$ I) *If $\mathcal{D}_1$ is a deduction proving $\Gamma \vdash A$, and $\mathcal{D}_2$ is a deduction proving $\Delta \vdash B$, and if (C−condition):$\Gamma \cap \Delta = \emptyset$ holds for, then*

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{A * B}\ (*I)$$

is a deduction proving $\Gamma \cup \Delta \vdash A * B$.

($*$ E) *If $\mathcal{D}_1$ is a deduction proving $\Gamma \vdash A \supset B \supset C$, and $\mathcal{D}_2$ is a deduction proving $\Delta \vdash A * B$, and if (C'−condition):$\Gamma \cap \Delta = \emptyset$ holds for, then*

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_2}{C}\ (*E)$$

is a deduction proving $\Gamma \cup \Delta \vdash C$.

Definition 3.3 (System NILL−W) *System NILL−W is a formal system obtained by removing (W−condition) from the system NILL.*

Definition 3.4 (System NILL−C) *System NILL−C is a formal system obtained by removing (C−condition) and (C'−conditon) from the system NILL.*

Definition 3.5 (System NILL−CW) *System NILL−CW is a formal system obtained by removing (C−condtion), (C'−condition) and (W−condition) from the system NILL.*

$$\begin{array}{c} \mathrm{W}{:}A^n \in \Gamma \\ \Gamma - \{A^n\} \\ \vdots \\ \dfrac{B}{A \supset B}\ (\supset I)^n \end{array} \qquad\qquad \begin{array}{c} \mathrm{C}{:}\Gamma \cap \Delta = \emptyset \\ \Gamma \quad\quad \Delta \\ \vdots \quad\quad \vdots \\ \dfrac{A \supset B \quad A}{B}\ (\supset E) \end{array}$$

$$\begin{array}{c} \mathrm{W}{:}\Gamma = \Delta \\ \Gamma \quad \Delta \\ \vdots \quad \vdots \\ \dfrac{A \quad B}{A \wedge B}\ (\wedge I) \end{array}$$

$$\begin{array}{c} \mathrm{C}{:}\Gamma \cap \Delta = \emptyset \\ \Gamma \quad\quad\quad \Delta \\ \vdots \quad\quad\quad \vdots \\ \dfrac{A \supset C \quad A \wedge B}{C}\ (\wedge E_l) \end{array} \qquad\qquad \begin{array}{c} \mathrm{C}{:}\Gamma \cap \Delta = \emptyset \\ \Gamma \quad\quad\quad \Delta \\ \vdots \quad\quad\quad \vdots \\ \dfrac{B \supset C \quad A \wedge B}{C}\ (\wedge E_r) \end{array}$$

$$\begin{array}{c} \mathrm{C}{:}\Gamma \cap \Delta = \emptyset \\ \Gamma \quad \Delta \\ \vdots \quad \vdots \\ \dfrac{A \quad B}{A * B}\ (*I) \end{array} \qquad\qquad \begin{array}{c} \mathrm{C}{:}\Gamma \cap \Delta = \emptyset \\ \Gamma \quad\quad\quad\quad \Delta \\ \vdots \quad\quad\quad\quad \vdots \\ \dfrac{A \supset B \supset C \quad A * B}{C}\ (*E) \end{array}$$

Rules for System NILL$*$

(example)

$$\cfrac{\cfrac{A^0 \quad A^0}{A \wedge A}\,(\wedge I)}{A \supset A \wedge A}\,(\supset I)^0$$

is a deduction proving $\vdash A \supset A \wedge A$ in NILL$*$.

$$\cfrac{\cfrac{A^1 \quad A^0}{A \wedge A}\,(\wedge I)}{A \supset A \wedge A}\,(\supset I)^0$$

is a deduction proving $\{A^1\} \vdash A \supset A \wedge A$ in NILL$-$W or NILL$-$CW.

To show that each System ILL$*$ is equivalent to the paired System NILL$*$, at first we prove the following lemma.

Lemma 3.6 (exchanging natural number of assumption) *In the System NILL$*$, if $\mathcal{D}$ is a deduction proving $\Gamma \cup \{A^n\} \vdash B$ $(A^n \notin \Gamma)$, then there is a deduction $\mathcal{E}$ proving $\Gamma \cup \{A^m\} \vdash B$ $(A^m \notin \Gamma)$ such that it is same height to $\mathcal{D}$*

(proof) Induction on height of $\mathcal{D}$.

Theorem 3.7 (equivalence of ILL$*$ and NILL$*$) *Each of the System ILL$*$ is equivalent to the paired System NILL$*$, i.e. there is a deduction D proving $A_1, ..., A_k \vdash A$ in ILL$*$, iff there is a deduction $\mathcal{E}$ proving $\{A_1^{n_1}\} \cup \cdots \cup \{A_k^{n_k}\} \vdash A$ $(A_1^{n_1}, ..., A_k^{n_k}$ are distinct each other$)$ in NILL$*$.*

(proof) Induction on height of deductions.

At first, we will show that if there is a deduction $\mathcal{D}$ proving $\Gamma \vdash A$ in ILL$*$, then there is a deduction $\mathcal{E}$ proving $\Gamma' \vdash A$ in NILL$*$.

1. Case $\mathcal{D} \equiv \dfrac{\mathcal{D}_1 \quad \mathcal{D}_2}{\Gamma \vdash A}\,(Cut)$: By definition, $\Gamma = \Delta, \Pi$, and $\mathcal{D}_1$ is a deduction proving $\Delta \vdash B$, $\mathcal{D}_2$ is a deduction proving $B, \Pi \vdash A$. By I.H., there is a deduction $\mathcal{E}_1$ proving $\Delta' \vdash B$ and a deduction $\mathcal{E}_2$ proving $\{B^n\} \cup \Pi' \vdash A$. By previous lemma, there is a deduction $\mathcal{E}_2'$ proving $\{B^m\} \cup \Pi'' \vdash A$ $(\Delta' \cap (\{B^m\} \cup \Pi'') = \emptyset,\ B^m \notin \Pi'')$ such that it is same height to $\mathcal{E}_2$. let
$$\mathcal{E} \equiv \cfrac{\cfrac{\mathcal{E}_2'}{B \supset A}\,(\supset I)^m \qquad \mathcal{E}_1}{A}\,(\supset E),$$
and it is a deduction $\Delta' \cup \Pi' \vdash A$.

2. Case $\mathcal{D} \equiv \dfrac{\mathcal{D}_1}{A, \Gamma \vdash B}\,(We)$: By definition, $\mathcal{D}_1$ is a deduction proving $\Gamma \vdash B$. By I.H., there is a deduction $\mathcal{E}_1$ proving $\Gamma' \vdash B$. Take $A^n \notin \Gamma'$, and let
$$\mathcal{E} \equiv \cfrac{\cfrac{\mathcal{E}_1}{A \supset B}\,(\supset I)^n \qquad A^n}{B}\,(\supset E) \text{ or}$$
$$\mathcal{E} \equiv \cfrac{\cfrac{B^0}{B \supset B}\,(\supset I)^0 \quad \cfrac{\mathcal{E}_1 \quad A^n}{A \wedge B}\,(\wedge I)}{B}\,(\wedge E),$$
and it is a deduction satisfying the condition.

3. Case $\mathcal{D} \equiv \dfrac{\mathcal{D}_1}{A, \Gamma \vdash B}\,(Co)$: By definition, $\mathcal{D}_1$ is a deduction proving $A, A, \Gamma \vdash B$. By I.H., there is a deduction $\mathcal{E}_1$ proving $\Gamma' \cup \{A^n, A^m\} \vdash B$ $(A^n, A^m \notin \Gamma',\ n \neq m)$. Let
$$\mathcal{E} \equiv \cfrac{\cfrac{\mathcal{E}_1}{A \supset B}\,(\supset I)^n \qquad A^m}{B}\,(\supset E),$$
$$\mathcal{E} \equiv \cfrac{\cfrac{\mathcal{E}_1}{A \supset B}\,(\supset I)^m \quad \cfrac{A^n \quad A^n}{A \wedge A}\,(\wedge I)}{B}\,(\wedge E_l),$$

$$\mathcal{E} \equiv \dfrac{\dfrac{\mathcal{E}_1}{A \supset B}(\supset I)^m \quad \dfrac{A^n \quad A^n}{A \wedge A}(\wedge I)}{B}(\wedge E_r) \text{ or}$$

$$\mathcal{E} \equiv \dfrac{\dfrac{\dfrac{\mathcal{E}_1}{A \supset B}(\supset I)^n}{A \supset A \supset B}(\supset I)^m \quad \dfrac{A^n \quad A^n}{A * A}(*I)}{B}(*E),$$

and it is a deduction satisfying the condition.

4. Another case: straightforward

Next, we will show that if there is a deduction $\mathcal{E}$ proving $\Gamma \vdash A$ in NILL$*$, then there is a deduction $\mathcal{D}$ proving $\Gamma' \vdash A$ in ILL$*$.

1. Case $\mathcal{E} \equiv A^n$: Let $\mathcal{D} \equiv A \vdash A$, and it is a deduction satisfying the condition.

2. Case $\mathcal{E} \equiv \dfrac{\mathcal{E}_1}{A \supset B}(\supset I)^n$: By definition, $\mathcal{E}_1$ is a deduction proving $\Gamma \cup \{A^n\} \vdash B$ or $\Gamma \vdash B$ such that $A^n \notin \Gamma$. By I.H., there is a deduction (a) $\mathcal{D}_1$ proving $\Gamma, A \vdash B$ or (b) $\mathcal{D}'_1$ proving $\Gamma \vdash B$. In (b), let
$$\mathcal{D}_1 \equiv \dfrac{\dfrac{\mathcal{D}'_1}{A, \Gamma \vdash B}(We)}{\Gamma, A \vdash B}(Ex),$$
and $\mathcal{D} \equiv \dfrac{\mathcal{D}_1}{\Gamma \vdash A \supset B}(\supset R)$,
and $\mathcal{D}$ is a deduction satisfying the condition.

3. Case $\mathcal{E} \equiv \dfrac{\mathcal{E}_1 \quad \mathcal{E}_2}{B}(\supset E)$: By definition, $\mathcal{E}_1$ is a deduction proving $\Gamma \cup \Delta \vdash A \supset B$, and $\mathcal{E}_2$ is a deduction proving $\Delta \cup \Pi \vdash A$ $(\Gamma \cap \Pi = \emptyset)$. By I.H., there is a deduction $\mathcal{D}_1$ proving $\Gamma', \Delta' \vdash A \supset B$ and a deduction $\mathcal{D}_2$ proving $\Delta', \Pi' \vdash A$. Let
$$\mathcal{D} \equiv \dfrac{\dfrac{\mathcal{D}_2 \quad \dfrac{\mathcal{D}_1 \quad \dfrac{A \vdash A \quad B \vdash B}{A, A \supset B \vdash B}(\supset L)}{\Gamma', \Delta', A \vdash B}(Cut)}{\Gamma', \Delta', \Delta', \Pi' \vdash B}(Cut)}{(\Gamma', \Delta', \Pi' \vdash B)}(Co),$$
and it is a deduction satisfying the condition.

4. Another case: similarly

Corollary 3.8 *The systems obtained by removing one of (W−condition) are equivalent to System ILL−W.*

Corollary 3.9 *The systems obtained by removing some of (C−condition) are equivalent to System ILL−C.*

Corollary 3.10 *The systems obtained by removing one of (W−condition) and some of (C−condition) are equivalent to System ILL−CW.*

(proof) Look proof of previous theorem carefuly, and these corollarys are obtained.

Corollary 3.11 *The system obtained by removing (C'−condition) and one of (W−condition) is equivalent to System ILL−CW*

(proof) It is enough to show that if $\Gamma \cup \{A^n\} \cup \{A^m\} \vdash B$ is provable, then $\Gamma \cup \{A^n\}$, in the system.

Take C^i and $(C \supset C)^k$ such that $C^i, (C \supset C)^k \notin \Gamma \cup \{A^n\} \cup \{A^m\} \vdash B$, let

$$\mathcal{D} \equiv \cfrac{\cfrac{\cfrac{\begin{matrix}\Gamma, A^n, A^m \\ \vdots \\ B\end{matrix}}{A \supset B}\,(\supset I)^m}{(C \supset C) \supset A \supset B}\,(\supset I)^k \quad \cfrac{\cfrac{C^i}{C \supset C}\,(\supset I)^i \quad A^n}{(C \supset C) * A}\,(\supset I)^k}{B}\,(*E)\ ,$$

and it is a deduction satisfying the condition.

4 Redex and reduction for a deduction of System NILL$*$

Next, like Gentzen's natural deduction systems NK and NJ, we define normal deduction. To define this, we define some difinitions, and prove some lemmas.

Definition 4.1 (substitution for deductions) *Let $\mathcal{E}$ be a deduction proving $\Gamma \vdash A$, and A^n be an assumption. We write $[\mathcal{E}/A^n]\mathcal{D}$ for substituting A^n in $\mathcal{D}$ proving $\Delta \vdash B$ by $\mathcal{E}$ and we define inductively on height of $\mathcal{D}$.*

1. *Case $\mathcal{D} \equiv A^n$: $[\mathcal{E}/A^n]A^n \equiv \mathcal{E}$*
2. *Case $\mathcal{D} \equiv A^m$ $(n \neq m)$: $[\mathcal{E}/A^n]A^m \equiv A^m$*
3. *Case $\mathcal{D} \equiv B^m$ $(B \not\equiv A)$: $[\mathcal{E}/A^n]B^m \equiv B^m$*
4. *Case $\mathcal{D} \equiv \dfrac{\mathcal{D}_1}{A \supset C}\,(\supset I)^n$: $[\mathcal{E}/A^n]\mathcal{D} \equiv \mathcal{D}$*
5. *Case $\mathcal{D} \equiv \dfrac{\mathcal{D}_1}{D \supset C}\,(\supset I)^m$ $(D \not\equiv A)$ and $D^m \notin \Gamma$, or $A^m \notin \Delta'$ where Δ' is the set of assumption of $\mathcal{D}_1$: $[\mathcal{E}/A^n]\mathcal{D} \equiv \dfrac{[\mathcal{E}/A^n]\mathcal{D}_1}{D \supset C}\,(\supset I)^m$*
6. *Case $\mathcal{D} \equiv \dfrac{\mathcal{D}_1}{D \supset C}\,(\supset I)^s$ $(D \not\equiv A)$, and $A^m \in \Gamma$ and if $\mathcal{D}_1$ proves $\Delta' \vdash C$ then $A^m \in \Delta'$: Take assumption D^k satisfying $D^k \notin \Delta' \cup \Gamma$, $[\mathcal{E}/A^n]\mathcal{D} \equiv \dfrac{[\mathcal{E}/A^n]([D^k/D^m]\mathcal{D}_1)}{A \supset B}\,(\supset I)^k$*
7. *Case $\mathcal{D} \equiv \dfrac{\mathcal{D}_1 \quad \mathcal{D}_2}{B}\,(R)$ $((R) = (\supset E), (\wedge I), (\wedge E_l), (\wedge E_r), (*I), (*E))$: $[\mathcal{E}/A^n]\mathcal{D} \equiv \dfrac{[\mathcal{E}/A^n]\mathcal{D}_1 \quad [\mathcal{E}/A^n]\mathcal{D}_2}{B}\,(R)$*

Lemma 4.2 *Let $\mathcal{D}$ be a deduction proving $\Delta \vdash B$, and $\mathcal{E}$ be one proving $\Gamma \vdash A$ in NILL$*$. If $A^n \notin \Delta$, then $[\mathcal{E}/A^n]\mathcal{D} \equiv \mathcal{D}$.*

(Proof) Induction on height of deductions.

Lemma 4.3 *Let $\mathcal{D}$ be a deduction proving $\Gamma \cup \{A^n\} \vdash B$ $(A^n \notin \Gamma)$, and $\mathcal{E}$ be a deduction proving $\Delta \vdash A$ in NILL$*$. If $\Gamma \cap \Delta = \emptyset$, then $\mathcal{F} \equiv [\mathcal{E}/A^n]\mathcal{D}$ is a deduction proving $\Gamma \cup \Delta \vdash B$ in it.*

(Proof) Induction on height of deductions.

Definition 4.4 (redex) *Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ be deductions in NILL$*$. If they are following forms, we call them* redex.

$$\cfrac{\cfrac{\mathcal{D}_1}{A \supset B}\,(\supset I)^n \quad \mathcal{D}_2}{B}\,(\supset E) \qquad \cfrac{\mathcal{D}_1 \quad \cfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{A \wedge B}\,(\wedge I)}{C}\,(\wedge E_l) \qquad \cfrac{\mathcal{D}_1 \quad \cfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{A \wedge B}\,(\wedge I)}{C}\,(\wedge E_r) \qquad \cfrac{\mathcal{D}_1 \quad \cfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{A * B}\,(*I)}{C}\,(*E)$$

Definition 4.5 (reduction) *we call it* reduction *to replace a left-side deduction by the right-side one.*

$$\dfrac{\dfrac{\mathcal{D}_1}{A \supset B}\,(\supset I)^n \quad \mathcal{D}_2}{B}\,(\supset E) \;\triangleright\; [\mathcal{D}_2/A^n]\mathcal{D}_1 \qquad \dfrac{\mathcal{D}_1 \quad \dfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{A * B}\,(*I)}{C}\,(*E) \;\triangleright\; \dfrac{\dfrac{\mathcal{D}_1 \quad \mathcal{D}_2}{B \supset C}\,(\supset E) \quad \mathcal{D}_3}{C}\,(\supset E)$$

$$\dfrac{\mathcal{D}_1 \quad \dfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{A \wedge B}\,(\wedge I)}{C}\,(\wedge E_l) \;\triangleright\; \dfrac{\mathcal{D}_1 \quad \mathcal{D}_2}{C}\,(\supset E) \qquad \dfrac{\mathcal{D}_1 \quad \dfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{A \wedge B}\,(\wedge I)}{C}\,(\wedge E_r) \;\triangleright\; \dfrac{\mathcal{D}_1 \quad \mathcal{D}_3}{C}\,(\supset E)$$

If $\mathcal{D} \triangleright \mathcal{D}'$, then we also call it reduction to replace a following left-side deduction by the right-side deduction.

$$\dfrac{\mathcal{D}}{A}\,(\supset I)^n \;\triangleright\; \dfrac{\mathcal{D}'}{A}\,(\supset I)^n$$

$$\dfrac{\mathcal{D} \quad \mathcal{E}}{A}\,(R) \;\triangleright\; \dfrac{\mathcal{D}' \quad \mathcal{E}}{A}\,(R) \qquad \dfrac{\mathcal{E} \quad \mathcal{D}}{A}\,(R) \;\triangleright\; \dfrac{\mathcal{E} \quad \mathcal{D}'}{A}\,(R)$$

$$(R) = (\supset E), (\wedge I), (\wedge E)_l, (\wedge E)_r, (*I), (*E)$$

Lemma 4.6 *Let $\mathcal{D}$ be a deduction proving $\Gamma \vdash A$ in NILL or NILL−C. If $\mathcal{D} \triangleright \mathcal{E}$, then $\mathcal{E}$ is a deduction proving $\Gamma \vdash A$ in it, too.*

(Proof) Induction on height of D.

Lemma 4.7 *Let $\mathcal{D}$ be a deduction proving $\Gamma \vdash A$ in NILL−W or NILL−WC. If $\mathcal{D} \triangleright \mathcal{E}$, then $\mathcal{E}$ is a deduction proving $\Gamma' \vdash A$ ($\Gamma' \subset \Gamma$) in it.*

(Proof) Induction on height of $\mathcal{D}$.

When we have normal proof in a system, even if we reduce a deduction in any way, we say that the system is strong normalizable. To show that NILL∗ are strong normalizable, we consider expanded typed linear∗ λ-term.

5 Expanded typed linear∗ λ-term

Definition 5.1 (typed linear λ-term) *Assume that there are enumerate infinite variables.*

1. x : *var*, A : *type* $\Rightarrow x^A$: *typed var*
2. x^A : *typed var* $\Rightarrow x^A$: *typed term*
3. x^A : *typed var*, M^B : *typed term*, *(W−condition)*:$x^A \in \mathrm{FV}(M^B)$
   $\Rightarrow (\lambda x^A.M^B)^{A \supset B}$: *typed term*
4. $M^{A \supset B}, N^A$: *typed term*, *(C−condition)*:$\mathrm{FV}(M^{A \supset B}) \cap \mathrm{FV}(N^A) = \emptyset$
   $\Rightarrow (M^{A \supset B} N^A)^B$: *typed term*
5. M^A, N^B : *typed term*, *(W−condition)*:$\mathrm{FV}(M^A) = \mathrm{FV}(N^B)$
   $\Rightarrow \langle M^A, N^B \rangle^{A \wedge B}$: *typed term*
6. $M^{A \supset C}, N^{A \wedge B}$: *typed term*, *(C−condition)*:$\mathrm{FV}(M^{A \supset C}) \cap \mathrm{FV}(N^{A \wedge B}) = \emptyset$
   $\Rightarrow (M^{A \supset C} \circ_l N^{A \wedge B})^C$: *typed term*
7. $M^{B \supset C}, N^{A \wedge B}$: *typed term*, *(C−condition)*:$\mathrm{FV}(M^{B \supset C}) \cap \mathrm{FV}(N^{A \wedge B}) = \emptyset$
   $\Rightarrow (M^{B \supset C} \circ_r N^{A \wedge B})^C$: *typed term*

8. M^A, N^B : *typed term, (C–condition):* $\mathrm{FV}(M^A) \cap \mathrm{FV}(N^B) = \emptyset$
   $\Rightarrow [M^A, N^B]^{A*B}$: *typed term*
9. $M^{A\supset(B\supset C)}, N^{A*B}$: *typed term, (C–condition):* $\mathrm{FV}(M^{A\supset(B\supset C)}) \cap \mathrm{FV}(N^{A*B}) = \emptyset \Rightarrow$ $(M^{A\supset(B\supset C)} \circ N^{A*B})^C$: *typed term*

We call the structure $[,]\,\langle,\rangle$ 'pairing' and $()\,(,)\,(\circ_l)\,(\circ_r)$ 'application'.

Definition 5.2 (typed linear–W λ-term) *Typed linear– W λ-term is defined by removing (W–condition) from typed linear λ-term.*

Definition 5.3 (typed linear–C λ-term) *Typed linear– C λ-term is defined by removing (C–condition) from typed linear λ-term.*

Definition 5.4 (typed linear–WC λ-term) *Typed linear– WC λ-term is defined by removing (W–condition) and (C–condition) from typed linear λ-term.*

Definition 5.5 (reduction) *we call it* reduction *to replace a following left-side term to the right-side term.*

1. $((\lambda x.M)^{A\supset B} N^A)^B \rhd [N^A/x^A]M^B$
2. $(M^{A\supset C} \circ_l \langle N^A, L^B\rangle^{A\wedge B})^C \rhd (M^{A\supset C} N^A)^C$
3. $(M^{B\supset C} \circ_r \langle N^A, L^B\rangle^{A\wedge B})^C \rhd (M^{B\supset C} L^B)^C$
4. $(M^{A\supset(B\supset C)} \circ [N^A, L^B]^{A*B})^C \rhd ((M^{A\supset(B\supset C)} N^A)^{B\supset C} L^B)^C$
   If $M^A \rhd N^A$,
5. $(\lambda x.M)^{B\supset A} \rhd (\lambda x.N)^{B\supset A}$
6. $(M, L)^B \rhd (N, L)^B$, $(L, M)^B \rhd (L, N)^B$ *((,)is any pairing or application)*

6 Strong normalization for linear or linear–W λ-term

Definition 6.1 (complexity) Complexity *of term (CP:Term → Nat) is defined inductively on structure of term as following.*

$$\begin{aligned}
\mathrm{MP}(x) &= 2 \\
\mathrm{MP}([M,N]) &= \mathrm{MP}(M) \times \mathrm{MP}(N) \\
\mathrm{MP}(\langle M,N\rangle) &= \mathrm{MP}(M) + \mathrm{MP}(N) \\
\mathrm{MP}(\lambda x.M) &= \mathrm{MP}(M) \\
\mathrm{MP}((M \circ N)) &= (\mathrm{MP}(M)+1) \times (\mathrm{MP}(N)+1) \\
\mathrm{MP}((M,N)) &= \mathrm{MP}(M) \times \mathrm{MP}(N) \quad \text{(another case)}
\end{aligned}$$

Lemma 6.2 *If* $(\lambda x.M)N$ *is a linear or linear– W term, then*

$$\mathrm{MP}((\lambda x.M)N) - (\mathrm{MP}(N) - 1) > \mathrm{MP}([N/x]M).$$

(Proof) Induction on structure of M.

Lemma 6.3 *If* M *is a linear or linear– W term,*

$$M \rhd N \Longrightarrow \mathrm{MP}(M) > \mathrm{MP}(N)$$

(Proof) Induction on structure of M.

1. Case $M \equiv (\lambda x.P)Q$ and $N \equiv [Q/x]P$: By previous lemma, $\mathrm{MP}(M) - (\mathrm{MP}(Q)+1) > \mathrm{MP}(N)$. Therefore $\mathrm{MP}(M) > \mathrm{MP}(N)$.
2. Another Case: Straightforward.

Theorem 6.4 (Strong Normalization) *All reductions of any linear or linear– W λ-term are finite.*

(Proof) By using previous lemma.

7 Strong Normalization for typed linear−WC λ-term

Definition 7.1 (Complexty of types) *We define complexty of type A* CP(A) *inductively on structure of type as follows:*

1. A : *atrmic* $\Rightarrow \mathrm{CP}(A) = 1$
2. $\mathrm{CP}(A \supset B) = \mathrm{CP}(A) + \mathrm{CP}(B)$
3. $\mathrm{CP}(A \wedge B) = (\mathrm{CP}(A) + \mathrm{CP}(B)) \times 2$
4. $\mathrm{CP}(A * B) = (\mathrm{CP}(A) + \mathrm{CP}(B)) \times 2$

Definition 7.2 (reducible) *We define a set* RED_A *of typed* λ*-term having type A as follows:*

1. $M^A \in \mathrm{RED}_A$ (A : *atomic*) $\Leftrightarrow A : SN$
2. $M^{A \supset B} \in \mathrm{RED}_{A \supset B} \Leftrightarrow \forall N^A \in \mathrm{RED}_A((M N)^B \in \mathrm{RED}_B)$
3. $M^{A \wedge B} \in \mathrm{RED}_{A \wedge B} \Leftrightarrow \forall N^{A \supset A} \in \mathrm{RED}_{A \supset A}, \forall L^{B \supset B} \in \mathrm{RED}_{B \supset B}$
   $((N \circ_l M)^A \in \mathrm{RED}_A, (N \circ_r M)^B \in \mathrm{RED}_B)$
4. $M^{A * B} \in \mathrm{RED}_{A * B} \Leftrightarrow \forall N^{A \supset (B \supset A)} \in \mathrm{RED}_{A \supset (B \supset A)}, \forall L^{A \supset (B \supset B)} \in \mathrm{RED}_{A \supset (B \supset B)}$
   $((N \circ M)^A \in \mathrm{RED}_A, (L \circ M)^B \in \mathrm{RED}_B)$

Definition 7.3 (neutral) *If* M^A *is a typed variable or an application, then we call it neutral.*

Definition 7.4 *If* M^A *: is SN, then we write* $\nu(M^A)$ *for the length of the longest reduction path of* M^A.

Lemma 7.5 (reducible) *Any typed* λ*-term* M^A *satisfies the following conditions from (CR1) to (CR4).*

(CR1) $M^A \in \mathrm{RED}_A \Rightarrow M^A : SN$

(CR2) $M^A \in \mathrm{RED}_A, M^A \rhd M'^A \Rightarrow M'^A \in \mathrm{RED}_A$

(CR3) $\forall M$: *neutral*, $\forall M'^A(M^A \rhd M'^A \Rightarrow M'^A \in \mathrm{RED}_A) \Rightarrow M^A \in \mathrm{RED}_A$

(CR4) M^A*:neutral and n.f.* $\Rightarrow M^A \in \mathrm{RED}_A$

(Proof) Induction on the complexty of type A.

Lemma 7.6 $M^A \in \mathrm{RED}_A, N^B \in \mathrm{RED}_B \Rightarrow (M, N)^C \in \mathrm{RED}_C$ *((,) is any pairing or application)*

(Proof) By usual method.

Lemma 7.7 M^A *:typed* λ*-term,* $N_1^{B_1}, ..., N_n^{B_n} \in \mathrm{RED} \Rightarrow [N_1^{B_1}/x_1^{B_1}, ..., N_n^{B_n}/x_n^{B_n}]M^A \in \mathrm{RED}_A$

(Proof) Induction on structure of M^A.

Theorem 7.8 (Strong Normalization) *All typed linear− WC* λ*-term is strongly nomalizable.*

(Proof) In previous lemma, let $N_1 \equiv x_1, ..., N_n \equiv x_n$, and $M^A \in \mathrm{RED}_A$ for any typed λ-term M^A. Therefore, by (CR1), M^A : SN.

Corollary 7.9 (Strong Normalization) *All typed linear$*$ λ-term is strongly nomalizable.*

8 Strong Normalization for NILL*

In this section, we show that all deductions in NILL* are strong normalizable by using strong normalization for typed linear−* λ-term.

because term variables are enumerate infinite, there is a bijective function from natural numbers to term variables. By using the function, there is a bijective function from deductions to linear* λ-term. We call it DT.

Lemma 8.1 $\mathcal{D} \rhd \mathcal{D}' \Rightarrow \mathrm{DT}(\mathcal{D}) \rhd \mathrm{DT}(\mathcal{D})'$

(Proof) Induction on structure of reduction.

Theorem 8.2 (Strong Normaliztion for NIIL*) *All deduction in NILL* are strongly nomalizable.*

(Proof) By using previous lemma.

9 Conclusion

- By attaching some indexes to assumptions, substraclural logics in natural deduction systems are introduced.
- All deductions in these systems are strongly normalizable.
- In the systems without contraction, there is a easy proof for strong nomalizations. To our interesting, additive conjunction pairing is defined by addition and multiplicative conjunction pairing is defined by multiplication.

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